Using Ratios of Successive Returns for the Estimation of Serial Correlation in Return Series

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Abstract: This paper proposes to estimate the first-order autocorrelation of asset returns by the rescaled sample mean of suitably transformed ratios of successive returns. The simplicity of this estimation method allows the monitoring of the stability of the estimates over time and the almost instantaneous detection of any structural break without any delay caused by an estimation window. In an empirical study of index returns, its use considerably increases the profitability of a simple trading strategy which switches between the index and cash.

Keywords: First-Order Autocorrelation, Structural Breaks, Trading Strategies.

1. Introduction

Estimating the correlation between successive daily stock returns is a difficult task for four reasons. First, the correlation is close to zero. Second, the distribution of the returns is fat-tailed. Third, large absolute returns occur in clusters. Fourth, the statistical properties of return series change over time. The problem is that all these unfavorable factors must be dealt with at the same time. Maximum likelihood (ML) estimation based on a fully specified model, e.g., a simple ARMA(1,0) model with Student $t$ GARCH(1,1) errors, is an obvious possibility. However, it would be unrealistic to expect that all specifications are correct; hence the maximum likelihood estimator will not be efficient. Moreover, estimation must be performed over rolling windows because of possible changes over time, which will further compromise accuracy. In general, more sophisticated models with a larger number of parameters will need longer estimation windows.

Fortunately, there are also facilitating factors. Since the mean of daily or intraday returns is extremely small compared to their standard deviation, it is usually preferable to set it to zero and introduce a small bias rather than to estimate it and significantly increase the variance. This is even true in the case of long return series because the assumption of a constant mean becomes increasingly implausible as the length of the observation period increases. Another favorable factor is that there is hardly any evidence of higher-order autocorrelation. Thus, the simplest autoregressive (AR) model

$$y_t = \rho y_{t-1} + \epsilon_t$$

with only one lag and no constant might suffice for the investigation of serial correlation in return series. Hurwicz (1950) proposed to use the median of the ratios

$$x_t = \frac{y_t}{y_{t-1}}$$

of successive observations as an estimator of $\rho$. Zielinski (1999) proved that Hurwicz’s estimator

$$\hat{\rho}_H = med(x_1, ..., x_n)$$

is median-unbiased under the assumption of continuous and independent innovations. We may expect that estimators of $\rho$, which are based on ratios of the form (2), will, on the one hand, be highly robust against
heteroscedasticity but, on the other hand, also quite unstable. In the case of asset returns, the latter issue is aggravated by the fact that returns close to zero as well as extreme returns are more frequent than in the Gaussian case.

Reschenhofer (2017) found that in the presence of extreme values and clusters of high volatility, the estimator \( \hat{\rho}_H \) outperforms conventional estimators such as the ML estimator, the least squares (LS) estimator, and several bias-corrected versions of the LS estimator (Marriott and Pope, 1954; White, 1961) and, in addition, that a further improvement can be achieved by applying the stabilizing transformation

\[
z_t = \begin{cases} 
  x_t, & \text{if } |x_t| \leq 1, \\
  \frac{1}{x_t}, & \text{if } |x_t| > 1
\end{cases} 
\]  

(4)

to the ratios \( x_t \). The distribution of \( z_t \) has the advantage that all its moments exist because it is defined over the finite interval \([-1,1]\), which implies in particular that the sample mean can be used instead of the sample median for the estimation of the parameter \( \rho \). However, since the sample mean \( \bar{z} \) is a severely biased estimator for \( \rho \), a bias correction is necessary.

In the next section, it is shown that a sufficient bias reduction can be achieved just by multiplying the sample mean \( \bar{z} \) by a constant. The only requirement is that \( \rho \) is not too large, which is usually the case in financial applications. Section 3 demonstrates the usefulness of the new estimator for the quick detection of structural breaks as well as for the construction of profitable trading strategies. Section 4 concludes.

2. A Simple Robust Estimator of First-Order Autocorrelation

If \( y_1 \) and \( y_{t-1} \) are two jointly normal variables satisfying

\[
E(y_1) = E(y_2) = 0
\]

(5)

and

\[
\text{var}(y_1) = \text{var}(y_2),
\]

(6)

then the ratio \( x_t \) defined by (2) has a Cauchy distribution with density

\[
f(x; \rho) = \frac{1}{\pi \theta \sqrt{(x-\rho)^2 + \theta^2}} = \frac{\sqrt{1-\rho^2}}{\pi} \frac{1}{x^2 - 2\rho x + 1},
\]

(7)

where the location parameter

\[
\rho = \text{corr}(Y_1, Y_2)
\]

(8)

and the scale parameter

\[
\theta = \sqrt{1 - \rho^2}
\]

(9)

specify the median and the interquartile range, respectively (see Jamnik (1971), for the distribution of more general ratios see Cedilnik et al. (2004); Marsaglia (2006)). Furthermore, the variable \( z_t \) defined by (4) has a truncated Cauchy distribution (see Nadarajah and Kotz (2006)) with density

\[
f^*(z; \rho) = 2f(x; \rho) = 2 \frac{\sqrt{1-\rho^2}}{\pi} \frac{1}{z^2 - 2\rho z + 1}, \quad -1 \leq z \leq 1
\]

(10)

and mean

\[
h(\rho) = \rho + \frac{\sqrt{1-\rho^2}}{\pi} \log \left( \frac{1-\rho}{1+\rho} \right)
\]

(11)

(see Reschenhofer (2017)). An approximately unbiased estimator for \( \rho \) is therefore given by

\[
\hat{\rho} = h^{-1}(\bar{z}),
\]

(12)
where $h^{-1}$ is the inverse function of $h$, which can be determined numerically to any desired precision.

Observing that in the empirically most relevant case, where $|\rho| \leq 0.2$, the functions $h$ and $h^{-1}$ are roughly linear (see Figure 1), it seems natural to approximate $h$ by its first-order Taylor series

$$\tilde{h}(\rho) = \frac{\pi-2}{\pi} \rho$$

(13)

and $h^{-1}$ by

$$\tilde{h}^{-1}(\xi) = \frac{\pi}{\pi-2} \xi.$$  

(14)

The resulting estimator

$$\tilde{\rho} = \tilde{h}^{-1}(\tilde{z}) = \frac{\pi}{\pi-2} \tilde{z}$$

(15)

will take a value close to $\rho$ whenever $\tilde{z}$ is close to $E(\tilde{z})$. For small values of $\rho$, $\tilde{\rho}$ will practically have the same properties as $\hat{\rho}$, which is known to be highly robust against heteroscedasticity and fat tails and outperforms $\hat{\rho}_c$ as well as conventional estimators under these assumptions. Apart from its simplicity, the new estimator $\tilde{\rho}$ has the unique advantage that

$$\tilde{\rho} = \tilde{h}^{-1}(\bar{z}) = \frac{1}{n} \sum_{t=1}^{n} \tilde{h}^{-1}(z_t),$$

(16)

which allows the almost instantaneous detection of structural breaks just by plotting the cumulative sums

$$\frac{1}{n} \sum_{t=1}^{j} \tilde{h}^{-1}(z_t) = \frac{\pi}{\pi-2} \frac{1}{n} \sum_{t=1}^{j} z_t, \quad j=1,\ldots,n,$$

(17)

against time. For any apparently stable subperiod $t_1,\ldots,t_2$, an estimate of the first-order autocorrelation can be obtained just by averaging the values

$$\frac{\pi}{\pi-2} Z_{t_1},\ldots,\frac{\pi}{\pi-2} Z_{t_2}.$$  

(18)

The next section presents the results obtained by applying the new estimator $\tilde{\rho}$ to a series of index returns. All computations are carried out with the free statistical software R (R Core Team, 2017).

**Figure 1.** Graphs of $h^{-1}$ (black) and the linear approximation $\tilde{h}^{-1}$ (gray)

### 3. Empirical Results

To illustrate the usefulness of the estimator proposed in the last section for the estimation of serial correlation in financial time series, the daily closing prices $P_t$ of the Standard & Poor's 500 Index (S&P
500) were downloaded from Yahoo! Finance. It is safe to assume that the first-order autocorrelation $\rho$ of today’s log returns

$$y_t = \log(P_t) - \log(P_{t-1})$$  \hspace{1cm} (19)$$
is small, hence the approximation (14) will be accurate enough and the simple estimator $\hat{\rho}$ will have similar properties as $\tilde{\rho}$.

Plotting the cumulative sums (17) against time, it can be seen that $\rho$ was positive in the 1960s and 1970s, close to zero in the 1980 and 1990s, and negative afterwards (see red line in Figure 2.a). The estimates obtained for these subperiods are $\tilde{\rho}=0.270$, 0.041, and $-0.094$, respectively. The estimates $\hat{\phi}_1=0.244$, 0.035, and $-0.051$, respectively, obtained by fitting an ARMA($p,q$)-GARCH($r,s$) model

$$y_t = \sum_{k=1}^{p} \phi_k y_{t-k} + \sum_{k=1}^{q} \theta_k u_{t-k} + u_t,$$  \hspace{1cm} (20)

with $t(3)$ innovations $\varepsilon_t$, where

- $p = 1, q = 0, r = s = 1,$
- $u_t = \sigma_t \varepsilon_t,$
- $\sigma_t^2 = var(u_t | \varepsilon_t, \varepsilon_{t-1}, \ldots) = \alpha_0 + \sum_{k=1}^{r} \alpha_k u_{t-k}^2 + \sum_{k=1}^{s} \beta_k \sigma_{t-k}^2,$

do not differ widely. In the latter case, estimation must be performed over rolling windows in order to detect changes over time and to identify stationary subperiods (see blue line in Figure 2.a; length of window: 250). However, depending on the length of the estimation window, there will be a certain time lag before a change becomes noticeable. For illustration, an image detail of Figure 2.a is shown in Figure 2.b. While the red line is flat in 1976, the blue line still has a positive slope. Plotting the cumulative sums

$$\frac{\sum_{t=1}^{j} y_t y_{t-1}}{\sum_{t=1}^{n} y_t^2}, \; j = 1, \ldots, n, \hspace{1cm} (21)$$

(green line in Figure 2.a) is also not an option because this procedure is very sensitive to (conditional) heteroscedasticity. Comparing Figures 2.a and 2.c, we see that the most extreme slopes, which are highly implausible, emerge just in periods of high volatility.

The greatest discrepancy between $\tilde{\rho}$ and $\hat{\phi}_1$ occurs in the third subperiod. To find out which of the two competing estimators is more reliable, a simple trading strategy is applied to the log index returns. Using an estimate of $\rho$ obtained from an estimation window of length 250, this strategy switches between the index and cash as follows. If the estimate of $\rho$ is positive/negative, the index is bought/sold at the end of a positive trading day and sold/bought at the end of a negative trading day. Figure 2.d shows that it does not matter before the beginning of the new millennium whether $\tilde{\rho}$ or $\hat{\phi}_1$ is used for the estimation of $\rho$. However, in the last subperiod, using $\tilde{\rho}$ is more profitable than using $\hat{\phi}_1$ (see Figure 2.d). Overall, this simple trading strategy clearly outperforms the buy-and-hold strategy (of course with the exception of the 1990s).

For a fair assessment of the performance of this strategy in the past, historical trading costs must be taken into account (see, e.g., Do and Faff (2012)). However, this is less relevant for today’s performance because discount brokers are now offering flat fee trading where a fixed amount is charged per trade. Moreover, the trading frequency can easily be reduced by looking not only at the sign of the log returns and the estimates of $\rho$ but also at their size and introducing suitable threshold values. However, as pointed out by Reschenhofer and Sinkovics (2017), we must be aware of the danger of data snooping whenever we increase the number of tuning parameters.
4. Concluding Remarks

A simple estimator of the first-order autocorrelation $\rho$ has been proposed, which is based on ratios of successive observations and is therefore robust against (conditional) heteroscedasticity by construction. Robustness against extreme values is achieved by replacing each ratio by its inverse whenever it is greater than one. Another important advantage of the new estimator is its simplicity. It is given by the sample mean of the transformed ratios multiplied by a constant that has been derived under the assumption of normality. The purpose of this constant is to reduce the bias. However, this bias correction works only if $\rho$ is small, which is usually the case in financial applications.

The results of an empirical study of index returns show that the simplicity of the estimator allows the monitoring of the stability of the estimates over time and the almost instantaneous detection of any structural break without any delay caused by an estimation window. Moreover, the results suggest that it may also be useful for the construction of new trading strategies or the improvement of existing trading strategies.

Tasks for future research include the application to financial high-frequency data and the extension to cross-correlation.

References


